ESTIMATING THE INSTABILITIES OF N CLOCKS BY MEANS OF COMPARISON MEASUREMENTS.

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Abstract

The estimation of individual instabilities of N clocks, compared by measuring the differences of their readings, is here considered without assuming a priori any hypotheses on their uncorrelation. Instabilities of the N clocks are described by a complete (non-diagonal) $N \times N$ covariance matrix R. Only differences of clock readings are available in order to estimate R. Statistical processing of these data allows one to calculate the $(N-1) \times (N-1)$ covariance matrix S of the differences relative to the N-th(reference) clock. By analyzing the relationships tying R and S, several pieces of information can be inferred and, in particular, the conditions for the validity of the uncorrelation hypothesis are established. The estimation of R from S is not unique: in any case R must be positive definite. A theorem states that R is positive definite if and only if its determinant is positive. Nevertheless infinitely many acceptable choices of R still fulfil the condition of positive definiteness. This paper shows that, by increasing the number N of compared clocks, the amount of arbitrariness in estimating R is reduced. The analysis of some experimental data illustrates the capability of the method.

1 INTRODUCTION

The evaluation of frequency standard instability is often performed by comparing N clocks and measuring differences in their signals. Results depend on the simultaneous contributions of all N standards and it is often desirable to estimate the noise contributions of any single units. In the past years, this problem has been considered in several papers, which introduced the popular "3–cornered hat" method [1], successively investigated and extended to N clocks [2, 3, 4, 5, 6].

The application of these methods often results in an unpleasant situation: some estimated clock variances turn out to be negative, violating the positiveness restriction intrinsic to their definition.

The assumption of uncorrelation of clocks, necessary in the above methods, seems to be too stringent and not always supported by experimental evidence ([7], see also [8]). Moreover, when more than 3 clocks are compared, the uncorrelation hypothesis leads to the formulation of an overdetermined linear system of equations [2, 3, 6, 9] and this overdetermination seems incompatible with the inconsistency of some results.

Here a new, consistent and more general mathematical model is proposed, suited for statistical processing of measured differences. The entire NxN covariance matrix $\mathbf{R}$ of the clocks along with the (N-1)x(N-1) covariance matrix $\mathbf{S}$ of the differences are introduced. Their relationship, and the possibility to estimate $\mathbf{R}$ from the knowledge of $\mathbf{S}$, are then discussed. A diagonal structure of $\mathbf{R}$ can result, a posteriori, from the most desirable situation of truly uncorrelated clocks. This model abides by the CIPM recommendation [10] in which it is recognized that "where appropriate, the estimated covariances should be given".

The examination of matrix $\mathbf{S}$ helps understanding the validity of the uncorrelation hypothesis. When such hypothesis is acceptable, the problem can be reduced to the classical case of the N-cornered hat, while in the other cases, the complete matrix $\mathbf{R}$ must be solved. In this second case the problem is underdetermined because the number of unknowns exceeds the number of equations. In any case there is an important constraint which binds the domain of acceptable solutions: the matrix $\mathbf{R}$, as any covariance matrix, must be positive definite. A theorem, specifically devised, is here presented showing that $\mathbf{R}$ is positive definite if and only if its determinant is positive. This constraint has been geometrically interpreted giving an insight of the features of the solution domain.

The interesting case of an increasing number of compared clocks is then examined showing how a larger number of clocks reduces the solution domain of $\mathbf{R}$.

2 STATEMENT OF THE PROBLEM

Let us denote by x^i the process related to the i-th clock and x^i_m its realization at the instant t_m. If M consecutive equispaced samples of the process are considered, they can be arranged in the vector $\mathbf{x}^i = [x^i_1, x^i_2, \ldots, x^i_M]^T$, where superscript ^{T} denotes the transposition. The expected value $\bar{x}^i$ of process x^i can be estimated in terms of the elements of $\mathbf{x}^i$ as $\bar{x}^i = (1/M)(x^i_1 + x^i_2 + \ldots + x^i_M)$ and arranged into the M-vector $\bar{\mathbf{x}}^i = [\bar{x}^i, \bar{x}^i, \ldots, \bar{x}^i]^T$. The M samples of the N clock processes are then cast in the $M \times N$ matrix $\mathbf{X} = [\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N]$. Similarly, the $M \times N$ matrix $\bar{\mathbf{X}}$ is introduced as $\bar{\mathbf{X}} = [\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^N]$. With these definitions the covariance matrix of the processes x^i, $i = 1, 2, \ldots, N$ is estimated by $\mathbf{R}$:

$$\mathbf{R} = [1/(M-1)][\mathbf{X} - \bar{\mathbf{X}}]^T[\mathbf{X} - \bar{\mathbf{X}}] \tag{1}$$

The diagonal element r_{ii} ($r_{ii} > 0$) denotes the variance of the i-th clock, while, for $i \neq j$, r_{ij} denotes the covariance between the i-th and the j-th clock. In order to obtain a good estimation of the matrix $\mathbf{R}$, it is required that the number of samples M be much larger than the number N of clocks.

When clocks are compared by measuring differences between their readings, the available data are the time differences $y^{ij} = x^i - x^j$. Also y^{ij} is statistically characterized by its expected value $\bar{y}^{ij}$

and by its variance: the estimates of these values are tied to the estimates of the variances r_{ii} and r_{jj} of the two single clocks and of their covariance r_{ij}.

One of the clocks, for instance clock #N, is chosen as the reference, and it is compared at M different instants with clocks #1, #2,..., #(N-1) giving $(N-1)$ distinct time difference measure M-vectors $\mathbf{y}^{1N} = \mathbf{x}^1 - \mathbf{x}^N$, $\mathbf{y}^{2N} = \mathbf{x}^2 - \mathbf{x}^N$, ..., $\mathbf{y}^{(N-1),\ N} = \mathbf{x}^{(N-1)} - \mathbf{x}^N$. Under the assumptions of contemporary and noiseless measurements, the other possible measure vectors $\mathbf{y}^{ij}$ $(i, j \neq N)$ are redundant, since they can be obtained as linear combination of those involving clock #N and they don't add any information, as proved in [11].

The M samples of the $N-1$ clock differences are cast in the $M \times (N-1)$ matrix $\mathbf{Y} = [\mathbf{y}^{1N},\ \mathbf{y}^{2N}, \ldots,\ \mathbf{y}^{N-1,\ N}]$. Similarly the $M \times (N-1)$ matrix $\bar{\mathbf{Y}}$ is introduced as $\bar{\mathbf{Y}} = [\bar{\mathbf{y}}^{1N},\ \bar{\mathbf{y}}^{2N}, \ldots,\ \bar{\mathbf{y}}^{(N-1),\ N}]$, where $\bar{\mathbf{y}}^{iN}$ is the M-vector whose elements are coincident with $\bar{y}^{iN}$.

The processes y^{iN}, $(i = 1,\ 2, \ldots,\ N-1)$ are statistically characterized by the estimation of the $(N-1) \times (N-1)$ covariance matrix $\mathbf{S}$ defined as:

$$\mathbf{S} = [1/(M-1)][\mathbf{Y} - \bar{\mathbf{Y}}]^T[\mathbf{Y} - \bar{\mathbf{Y}}] \tag{2}$$

A generic element s_{ij} denotes either a variance (for $i = j$) and it is always positive or a covariance (for $i \neq j$) and may assume any real value. Note that the index N of the reference clock has been dropped in any element of $\mathbf{S}$.

The relationship between matrices $\mathbf{S}$ and $\mathbf{R}$ is easily determined by observing that $\mathbf{Y}$ can be derived from $\mathbf{X}$, according to the following relationship:

$$\mathbf{Y} = \mathbf{X}\,\mathbf{H} \tag{3}$$

where $\mathbf{H}$ is the $N \times (N-1)$ matrix:

$$\mathbf{H} = \begin{bmatrix} \hat{\mathbf{I}} \\ -\mathbf{u}^T \end{bmatrix} \tag{4}$$

where $\hat{\mathbf{I}}$ is the $(N-1) \times (N-1)$ identity matrix and $\mathbf{u}$ is the $(N-1)$-vector $[1,\ 1, \ldots,\ 1]^T$. A similar relationship ties the corresponding matrices $\bar{\mathbf{X}}$ and $\bar{\mathbf{Y}}$.

From (1), (2), and (3), the covariance matrix $\mathbf{S}$ can be expressed in terms of $\mathbf{R}$ as:

$$\mathbf{S} = \mathbf{H}^T\mathbf{R}\mathbf{H} \tag{5}$$

For sake of generality here we discuss the covariance matrices of the processes themselves, but the same properties hold also for any covariance matrix defined from prefiltered data as is generally the case in the estimation of clock instability, where the Allan variance is used.

3 CONSIDERATIONS ON R AND S

By taking into account the symmetry of $\mathbf{R}$ and $\mathbf{S}$ it appears that $\mathbf{R}$ is defined by $N(N-1)$ scalars and $\mathbf{S}$ by $(N-1)N/2$ ones, then the knowledge of $\mathbf{S}$ is not sufficient to fix a unique estimation

of $\mathbf{R}$ because ∞^N solutions satisfy (5) (underdetermined problem). A way to isolate the N free parameters consists in partitioning $\mathbf{R}$ according to:

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{R}} & \mathbf{r} \\ \mathbf{r}^T & r_{NN} \end{bmatrix} \qquad (6)$$

where $\hat{\mathbf{R}}$ is the leading $(N-1) \times (N-1)$ submatrix; $\mathbf{r} = [r_{1N},\ r_{2N}, \ldots,\ r_{N-1,\ N}]^T$ is the $(N-1)$-vector, grouping the covariances involving the N-th clock and r_{NN} is the variance of the N-th clock. With this partition, and by substituting (4), equation (5) can be rewritten as:

$$\mathbf{S} = [\hat{\mathbf{I}} - \mathbf{u}] \begin{bmatrix} \hat{\mathbf{R}} & \mathbf{r} \\ \mathbf{r}^T & r_{NN} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}} \\ -\mathbf{u}^T \end{bmatrix} = \hat{\mathbf{R}} + r_{NN}[\mathbf{u}\ \mathbf{u}]^T - \mathbf{u}\ \mathbf{r}^T - \mathbf{r}\ \mathbf{u}^T \qquad (7)$$

This equation shows that $\mathbf{R}$ can be uniquely reconstructed from $\mathbf{S}$ if the (co)variances related to the N-th clock, i.e. $\mathbf{r}$ and r_{NN}, are known. In fact, (7) can be transformed in the following expression:

$$\hat{\mathbf{R}} = \mathbf{S} - r_{NN}[\mathbf{u}\ \mathbf{u}^T] + \mathbf{u}\ \mathbf{r}^T + \mathbf{r}\ \mathbf{u}^T \qquad (8)$$

With a deeper look into (7), some qualitative information about $\mathbf{R}$ can be immediately deduced when $\mathbf{S}$ is known:

1. If the N clocks are uncorrelated, $\mathbf{S}$ assumes the form:

$$\mathbf{S} = \begin{bmatrix} r_{11} + r_{NN} & r_{NN} & \cdots & r_{NN} \\ r_{NN} & r_{22} + r_{NN} & \cdots & r_{NN} \\ \cdots & \cdots & \cdots & \cdots \\ r_{NN} & r_{NN} & \cdots & r_{N-1,\ N-1} + r_{NN} \end{bmatrix} \qquad (9)$$

   All the covariances s_{ij} (with $i \neq j$) are equal and positive, because $s_{ij} = r_{NN} > 0$. Moreover each variance $s_{ii} > s_{ij}$ (with $i \neq j$). These conditions are necessary to validate the assumption of uncorrelation of the N clocks. Unfortunately they are not sufficient to conclude that the clock are uncorrelated because, together with a unique diagonal solution, infinitely many other non–diagonal matrices $\mathbf{R}$ would drive to the same matrix $\mathbf{S}$, but if these conditions are satisfied one can "reasonably" assume the uncorrelation.

2. If the reference clock is "quasi–ideal" (that is $r_{NN} \ll r_{ii}$ for $i = 1,\ 2, \ldots,\ N-1$) and all the clocks are uncorrelated, then $s_{ij} \ll s_{ii}$ for any $i \neq j$ and $\mathbf{S}$, as well as $\mathbf{R}$, can be considered diagonal. So, if $\mathbf{S}$ is almost diagonal, the reference clock is of high quality. In this case, the submatrix $\hat{\mathbf{R}}$ almost coincides with $\mathbf{S}$ and $\mathbf{r} \cong 0$, $r_{NN} \cong 0$.

3. For the same reason, when the hypothesis of uncorrelation holds, if the reference clock is changed and $\mathbf{S}$ is computed again, it gives an idea of which one of the clocks is less noisy, because it results in a matrix $\mathbf{S}$ with minimum off–diagonal terms.

4. If any s_{ij} (with $i \neq j$) is negative, then the uncorrelation hypothesis is certainly to be excluded at least between the $(i,\ N)$-th or the $(j,\ N)$-th pair of clocks.

5. If all the terms s_{ij} (with $i \neq j$) are equal and positive except one, for instance s_{kl}(with $k \neq l$), which is positive but differs from the others, that indicates that all the clocks can be considered uncorrelated except the pair (k, l).

6. If the terms s_{ij} (with $i \neq j \neq k$) are coincident and positive, and the terms s_{ij} (with $i \neq j$, $i = k$, or $j = k$) are coincident but different from the previous ones, the k-th clock is possibly correlated with the reference one while all the others are uncorrelated.

From the theoretical point of view, the above considerations suggest clear interpretations. By handling experimental data, it can be difficult to recognize the points above because of round-off error, not perfect contemporaneity of measurement, low noise introduced by the measurement system but, above all, the low number of statistical samples that give a considerable uncertainty bar on the estimates.

4 POSITIVE DEFINITENESS OF S AND R

The estimation of the complete matrix **R** asks for a suitable choice on the N free parameters in (8). In any case, there is an important constraint which bounds the solution domain $\mathcal{D}$ in the space of the N free parameters and which guarantees a significant result: as any covariance matrix [13], *the estimated covariance matrix* **R** *must be positive definite.*

Some significant properties of positive definite covariance matrices are here reminded. Let **A** be an $N \times N$ symmetrical matrix and $\mathbf{A}^{(n)}$ ($n = 1, 2, \ldots, N$) be the leading $n \times n$ submatrices of **A**, extracted from **A** by considering the elements belonging to the first n rows and n columns (with $\mathbf{A}^{(N)} \equiv \mathbf{A}$). The matrix **A** is positive definite if and only if [13]:

$$|\mathbf{A}^n)| > 0 \quad \text{for } n = 1, 2, \ldots, N \tag{10}$$

where $| \, . \, |$ denotes the determinant.

Then the positive definiteness of **A** requires the validation of N scalar inequalities. In the case of **R** and **S**, since they are both positive definite and they are linked by (5), a theorem holds [11]:

Theorem 1: The $N \times N$ symmetrical matrix **R**, related to the known positive definite $(N-1) \times (N-1)$ symmetrical matrix **S** by the relationship (5), is also positive definite if and only if $|\mathbf{R}| > 0$.

This theorem allows one to verify the positive definiteness of **R** by considering only one of the inequalities (10); moreover this inequality can be geometrically interpreted. In fact, the relationship $|\mathbf{R}| = 0$ appears to be an equation of second degree in the $N - 1$ elements of **r** and in r_{NN}.

With the partition of **R** used in (6), it can be demonstrated [11] that its determinant can be expressed as:

$$|\mathbf{R}| = |\mathbf{S}|(r_{NN} - [\mathbf{r} - r_{NN}\mathbf{u}]^T\mathbf{S}^{-1}[\mathbf{r} - r_{NN}\mathbf{u}]) \tag{11}$$

with **u** defined in (4). The boundary of the solution domain $\mathcal{D}$ is therefore described by the surface $|\mathbf{R}| = 0$:

$$[\mathbf{r} - r_{NN}\mathbf{u}]^T\mathbf{S}^{-1}[\mathbf{r} - r_{NN}\mathbf{u}] = r_{NN} \tag{12}$$

If r_{NN} is fixed, this expression represents an ellipsoid in the $(N-1)$dimensional space described by the variables $r_{1N}, r_{2N}, \ldots, r_{N-1, N}$ because the matrix $\mathbf{S}^{-1}$ of the quadratic form is positive definite.

This ellipsoid is centered at the point of coordinates $(r_{1N} = r_{NN}, r_{2N} = r_{NN}, \ldots, r_{N-1, N} = r_{NN})$, the principal axes are rotated and the squared length of the axes is proportional to r_{NN}. As far as the parameter r_{NN} is concerned, the domain $\mathcal{D}$ takes the form of an elliptic hyper-paraboloid in the N-dimensional space. The intersection with a plane ($r_{NN} = $ constant) leads to an $(N-1)$-dimensional ellipsoid, the increase of r_{NN} increases the surface and shifts the center of the ellipsoid (12), and it doesn't rotate its axes. Fig. 1 shows an example of the domain $\mathcal{D}$ in the case of $N = 3$.

The constraint of positive definiteness of $\mathbf{R}$ bounds the domain of choice of the N free parameters $\mathbf{r}$ and r_{NN} and only the set of values of $r_{1N}, r_{2N}, \ldots, r_{N-1, N}, r_{NN}$ situated inside the surface described by (12) guarantees acceptable solutions.

5 EFFECT OF AN INCREASING NUMBER OF COMPARED CLOCKS

Knowing the expression of the boundary of the domain $\mathcal{D}$, it can be interesting to test how it is modified by varying the number of clocks involved in the measurements. Within a set of N clocks let's individuate the N-th clock as the reference one and two subset $\mathcal{C}^a$ of N^a clocks and $\mathcal{C}^b$ of N^b clocks. $\mathcal{C}$ denotes the union of $\mathcal{C}^a$ and $\mathcal{C}^b$ and $N = N^a + N^b + 1$. With the above notation, the $N \times N$ covariance matrix $\mathbf{R}$ of the N clocks is partitioned as follows:

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}^{aa} & \mathbf{R}^{ab} & \mathbf{r}^a \\ [\mathbf{R}^{ab}]^T & \mathbf{R}^{bb} & \mathbf{r}^b \\ [\mathbf{r}^a]^T & [\mathbf{r}^b]^T & r_{NN} \end{bmatrix} \tag{13}$$

where $\mathbf{R}^{aa}$ and $\mathbf{R}^{bb}$ are the covariance matrices related to the clocks in $\mathcal{C}^a$ and $\mathcal{C}^b$ respectively. $\mathbf{R}^{ab}$ contains the covariances between clocks in $\mathcal{C}^a$ and $\mathcal{C}^b$ versus the N-th clock; they represent a partition of the vector $\mathbf{r}$ introduced in (6). The element r_{NN} is the variance of the N-th clock.

Analogously, the $(N-1) \times (N-1)$ covariance matrix $\mathbf{S}$ of the differences can be partitioned as:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}^{aa} & \mathbf{S}^{ab} \\ [\mathbf{S}^{ab}]^T & \mathbf{S}^{bb} \end{bmatrix} \tag{14}$$

where $\mathbf{S}^{aa}$ and $\mathbf{S}^{bb}$ are the covariance matrices of the differences related to the clocks in $\mathcal{C}^a$ and $\mathcal{C}^b$ respectively.

If the clocks in $\mathcal{C}^a$ and $\mathcal{C}$ (jointly with the reference one) are considered, the respective solution domains $\mathcal{D}^a$ and $\mathcal{D}$ are:

$$[\mathbf{r}^a - r_{NN}\mathbf{u}^a]^T [\mathbf{S}^{aa}]^{-1} [\mathbf{r}^a - r_{NN}\mathbf{u}^a] \leq r_{NN} \tag{15}$$

$$[\mathbf{r} - r_{NN}\mathbf{u}]^T \mathbf{S}^{-1} [\mathbf{r} - r_{NN}\mathbf{u}] \leq r_{NN} \tag{16}$$

where $\mathbf{u}^a$ is the N^a-vector $[1, 1, \ldots, 1]^T$.

Expression (16) can be rewritten in terms of the submatrices of $\mathbf{S}$ in (14). In fact, the vector $\mathbf{r} - r_{NN}\mathbf{u}$ can be partitioned as:

$$[\mathbf{r} - r_{NN}\mathbf{u}] = \begin{bmatrix} \mathbf{r}^a - r_{NN}\mathbf{u}^a \\ \mathbf{r}^b - r_{NN}\mathbf{u}^b \end{bmatrix} \tag{17}$$

where $\mathbf{u}^b$ is the N^b-vector $[1, 1, \ldots, 1]^T$ and the inverse of $\mathbf{S}$ can be written as:

$$\mathbf{S}^{-1} = \begin{bmatrix} [\mathbf{S}^{aa}]^{-1} + [\mathbf{S}^{aa}]^{-1}\,\mathbf{S}^{ab}\,[\mathbf{D}^{bb}]^{-1}\,[\mathbf{S}^{ab}]^T\,[\mathbf{S}^{aa}]^{-1} & -[\mathbf{S}^{aa}]^{-1}\,\mathbf{S}^{ab}\,[\mathbf{D}^{bb}]^{-1} \\ -[\mathbf{D}^{bb}]^{-1}\,[\mathbf{S}^{ab}]^T\,[\mathbf{S}^{aa}]^{-1} & [\mathbf{D}^{bb}]^{-1} \end{bmatrix} \qquad (18)$$

where the matrix $\mathbf{D}^{bb}$ is defined as [12]:

$$\mathbf{D}^{bb} = \mathbf{S}^{bb} - [\mathbf{S}^{ab}]^T[\mathbf{S}^{aa}]^{-1}\mathbf{S}^{ab} \qquad (19)$$

Now the idea is to compare the extension of the domains delimited by the elliptic paraboloids $\mathcal{D}^a$ and $\mathcal{D}$, defined by (15) and (16), respectively. The domain $\mathcal{D}^a$ is defined in the space of N^a+1 variables $r_{1N}, r_{2N}, \ldots, r_{N^aN}, r_{NN}$, while $\mathcal{D}$ is defined in the space of N variables $r_{1N}, r_{2N}, \ldots, r_{N-1,N}, r_{NN}$. To the aim of comparison, it is necessary to individuate a sub-domain $\bar{\mathcal{D}}$ obtained from $\mathcal{D}$ by a suitable projection in the subspace of N^a+1 variables $r_{1N}, r_{2N}, \ldots, r_{N^aN}, r_{NN}$.

Domain $\bar{\mathcal{D}}$ is obtained by the projection of the intersection of $\mathcal{D}$ with the N^b-dimensional subspace defined by the hyperplane $r_{NN}\mathbf{u}^b = \mathbf{r}^b$. Since this choice corresponds to the largest intersection of $\mathcal{D}$, $\bar{\mathcal{D}}$ results in the largest possible subdomain in the (N^a+1)-dimensional subspace and therefore it yields the least favourable case in the comparison of the two domains.

By substituting (17), (18), (19) and $r_{NN}\mathbf{u}^b = \mathbf{r}^b$ in (16), the equation defining domain $\bar{\mathcal{D}}$ is obtained as:

$$[\mathbf{r}^a - r_{NN}\mathbf{u}^a]^T[\mathbf{S}^{aa}]^{-1}[\mathbf{r}^a - r_{NN}\mathbf{u}^a] + [\mathbf{r}^a - r_{NN}\mathbf{u}^a]^T\{[\mathbf{S}^{aa}]^{-1}\mathbf{S}^{ab}[\mathbf{D}^{bb}]^{-1}[\mathbf{S}^{ab}]^T[\mathbf{S}^{aa}]^{-1}\}[\mathbf{r}^a - r_{NN}\mathbf{u}^a] = r_{NN} \qquad (20)$$

By comparing $\bar{\mathcal{D}}$ (20) with $\mathcal{D}^a$ (15), it appears that the only difference is in the second quadratic form in the l.h.s. of (20). From the properties of the definite positive matrices, since $\mathbf{S}$ and its submatrix $\mathbf{S}^{aa}$ are positive definite, so are their inverses, and it can be demonstrated that also the matrix $[\mathbf{S}^{aa}]^{-1}\mathbf{S}^{ab}[\mathbf{D}^{bb}]^{-1}[\mathbf{S}^{ab}]^T[\mathbf{S}^{aa}]^{-1}$ of the second quadratic form in (20) is positive definite. So it necessarily results that $\bar{\mathcal{D}}$ is contained inside $\mathcal{D}^a$. As a result, an increased number of compared clocks reduces the solution domain and therefore the amount of arbitrariness in the determination of the covariance matrix $\mathbf{R}$, even in the worst considered case, i. e. with the largest possible intersection. It must be stressed that the reduction ratio of the solution domains depends only on the elements of matrix $\mathbf{S}$ and not on the value of r_{NN}. Intuitively, if the clocks in $\mathcal{C}^b$ are completely correlated with any clock in $\mathcal{C}^a$ or with the reference one, their measurement doesn't add any further information to the problem and the reduction of the solution domain will be very poor. On the contrary, the addition of independent clocks can result in a significant shrinking of the solution domain.

6 EXAMPLE

Five commercial cesium clocks have been compared. Three of the clocks are maintained at IEN, Turin, Italy (Cs4, Cs5, Cs6), the other two clocks are maintained at ISPT, Rome, Italy and Telespazio, Matera, Italy and data are regularly transmitted to the IEN for traceability. The comparison at distance is obtained with GPS or TV link. An appropriate smoothing procedure has been applied to reduce the additive noise due to the synchronization link and, for integration times larger than five days, this noise can be assumed to be negligible. The Allan covariance matrix

has been computed for different integration times, with overlapping samples, by using one year (1991) of daily measurement data. (The Allan covariance matrix is an extension of the well–known Allan variance in the sense of (1)). In the present example the set $\mathcal{C}^a$ is composed of clocks Cs4 and Cs5 of IEN, the set $\mathcal{C}^b$ is composed of the two external clocks and the clock Cs6 of IEN is the reference one. When the clocks in $\mathcal{C}^a$ jointly to the reference are used, the 2x2 matrix $\mathbf{S}^{aa}$ is obtained while, with the five clocks, the 4x4 matrix $\mathbf{S}$ is obtained.

To investigate how the increased number of clocks reduces the solution domain, $\bar{\mathcal{D}}$ is compared with $\mathcal{D}^a$ in the plane (r_{15}, r_{25}), with a fixed value for the variance r_{55}. Here we report only some of the most significant examined cases.

The first case regards the Allan covariance matrix evaluated for integration time $\tau = 10$ days. The following matrices $\mathbf{S}^{aa}$ and $\mathbf{S}$ are obtained, where numerical values have been scaled by 10^{28}:

$$\mathbf{S}^{aa} = \begin{bmatrix} 11.8 & 4.78 \\ 4.78 & 9.34 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 11.8 & 4.78 & 8.64 & 7.10 \\ 4.78 & 9.34 & 7.38 & 1.31 \\ 8.64 & 7.38 & 96.7 & -23.7 \\ 7.10 & 1.31 & -23.7 & 218 \end{bmatrix} \qquad (21)$$

Since matrix $\mathbf{S}^{aa}$ satisfies conditions (9), the uncorrelation hypothesis is acceptable for the clocks in $\mathcal{C}^a$ and the reference one. We can choose $r_{55} = s_{12}$, $r_{15} = 0$, and $r_{25} = 0$ driving to the classical solution. By choosing $r_{55} = s_{12}$, the two domains $\mathcal{D}^a$ and $\bar{\mathcal{D}}$ delimiting the possible choices for r_{15} and r_{25} are compared in Fig. 2. As expected, the classical solution corresponding to $r_{15} = 0$ and $r_{25} = 0$ is inside the domain $\mathcal{D}$, and it is also inside the domain $\bar{\mathcal{D}}$. It also means that from the comparisons of the five clocks it results that the clocks in $\mathcal{C}^a$ and the reference one may be considered uncorrelated. That is not the case for the other clocks because the complete matrix $\mathbf{S}$ has not all equal and positive off-diagonal elements.

From Fig.2 it can be verified that the addition of the (co)variances of the clocks in $\mathcal{C}^b$ slightly shrinks the solution domain and therefore also the arbitrariness in the choices of the free parameters r_{15} and r_{25} is reduced.

The second case concerns the instability estimation of the same clocks for $\tau = 100$ days. The following matrices $\mathbf{S}^{aa}$ and $\mathbf{S}$ are obtained:

$$\mathbf{S}^{aa} = \begin{bmatrix} 104 & 20.3 \\ 20.3 & 16.1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 104 & 20.3 & 97.4 & 103 \\ 20.3 & 16.1 & 19.8 & -41.6 \\ 97.4 & 19.8 & 97.3 & 88.9 \\ 103 & -41.6 & 88.9 & 433 \end{bmatrix} \qquad (22)$$

where the values are again scaled by 10^{28}.

Here the conditions (9) are not satisfied neither in the comparison of three clocks ($\mathbf{S}^{aa}$), nor with five clocks ($\mathbf{S}$). The solution domain of the covariances r_{15} and r_{25} is represented in Fig. 3. Here we see a significant reduction of $\bar{\mathcal{D}}$ with respect to $\mathcal{D}^a$ when five clocks are compared. The clocks in $\mathcal{C}^a$ add so much information that the range of acceptable values for r_{15} and r_{25} is reduced by a factor of 5. Inside the domain $\bar{\mathcal{D}}$, the arbitrariness in choosing r_{15} and r_{25} is so much reduced that it seems not so much crucial the criterion to be used for the final choice. This situation is also

depicted in Fig. 4 where the two 3–dimensional elliptic paraboloids describe the boundaries of the solution domains for r_{15}, r_{25}, and r_{55}.

7 CONCLUSIONS

This work has considered the estimation of clock instabilities when 3 or more clocks are compared by measuring signal differences. From the measure covariance matrix S several information about the instability of the individual clocks can be inferred and, in particular, the uncorrelation hypothesis can be accepted or rejected. The estimation of the individual clock covariance matrix R from the knowledge of S is an underdetermined problem and an appropriate optimization criterion ought to be formulated to fix the remaining free parameters, but the solution domain is constrained by the request of positive definiteness for the resulting R. Such a constraint has been deeply analyzed and, with a theorem, it has been possible to give a simply geometrical illustration. Moreover the importance of the comparison of a larger number of clocks is outlined, showing how, in some cases, it results in a significant reduction of the arbitrariness in estimating clock instability.

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Fig.1: Solution domain $\mathcal{D}$ corresponding to a particular matrix S obtained from the comparison of 3 different clocks.
(a) the domain $\mathcal{D}$ in the space of variables r_{13}, r_{23}, and r_{33}.
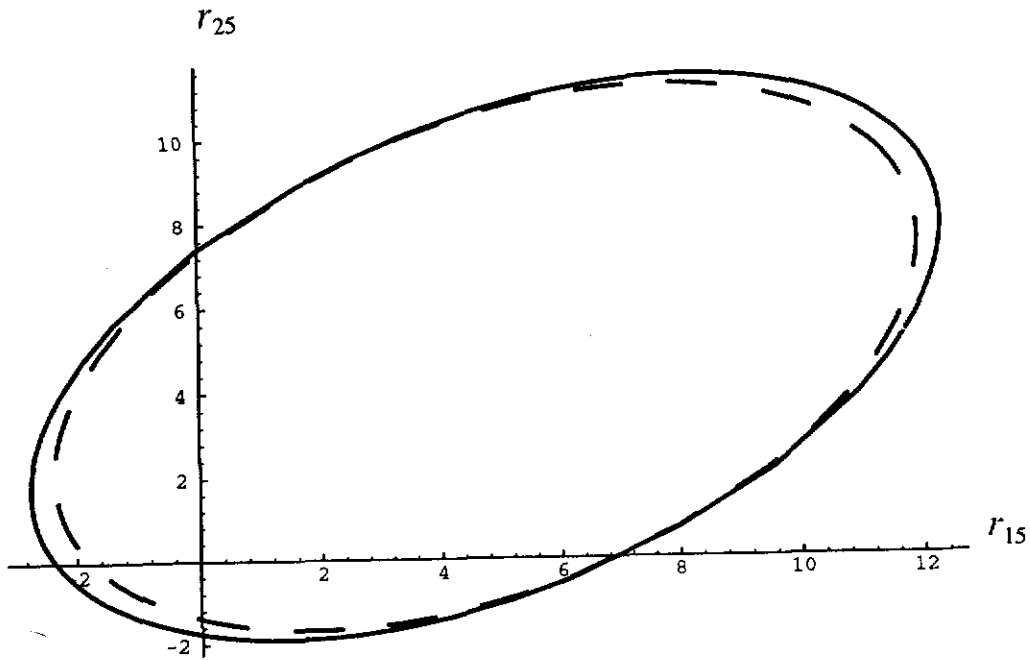(b) boundaries of $\overline{\mathcal{D}}$ for some fixed values of r_{33}.

Fig.2: Domains $\mathcal{D}'$ (solid line) and $\overline{\mathcal{D}}$ (dashed line) in the plane (r_{15}, r_{25}) for integration time $\tau = 10$ days and $r_{55} = 4.78*10^{-28}$.
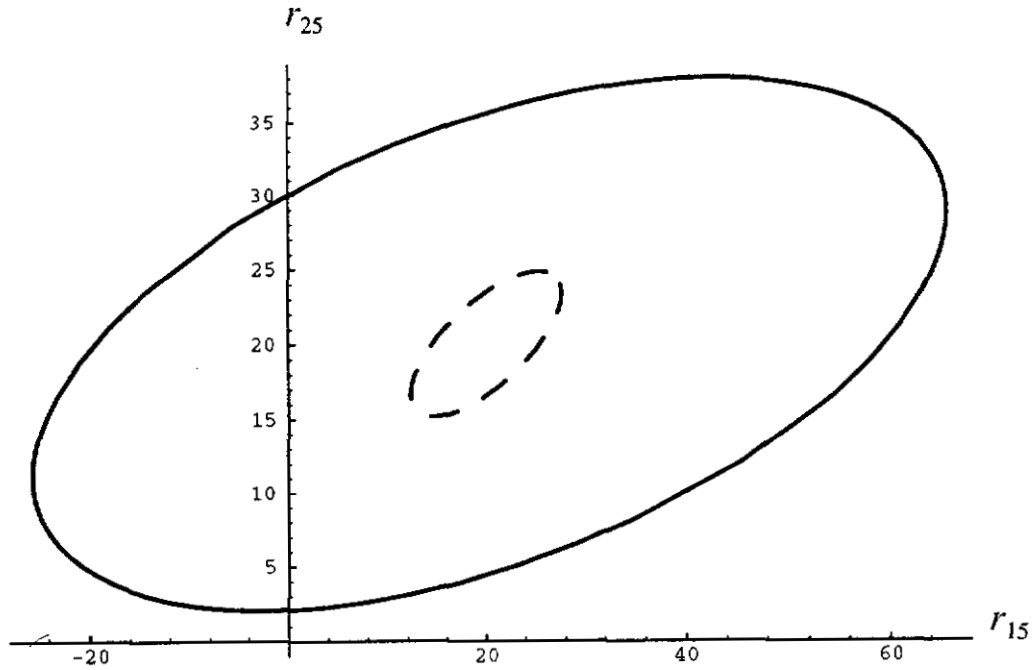
Fig.3: Domains $\mathcal{D}^i$ (solid line) and $\overline{\mathcal{D}}$ (dashed line) in the plane (r_{15}, r_{25}) for integration time $\tau = 100$ days and $r_{55} = 20*10^{-28}$.
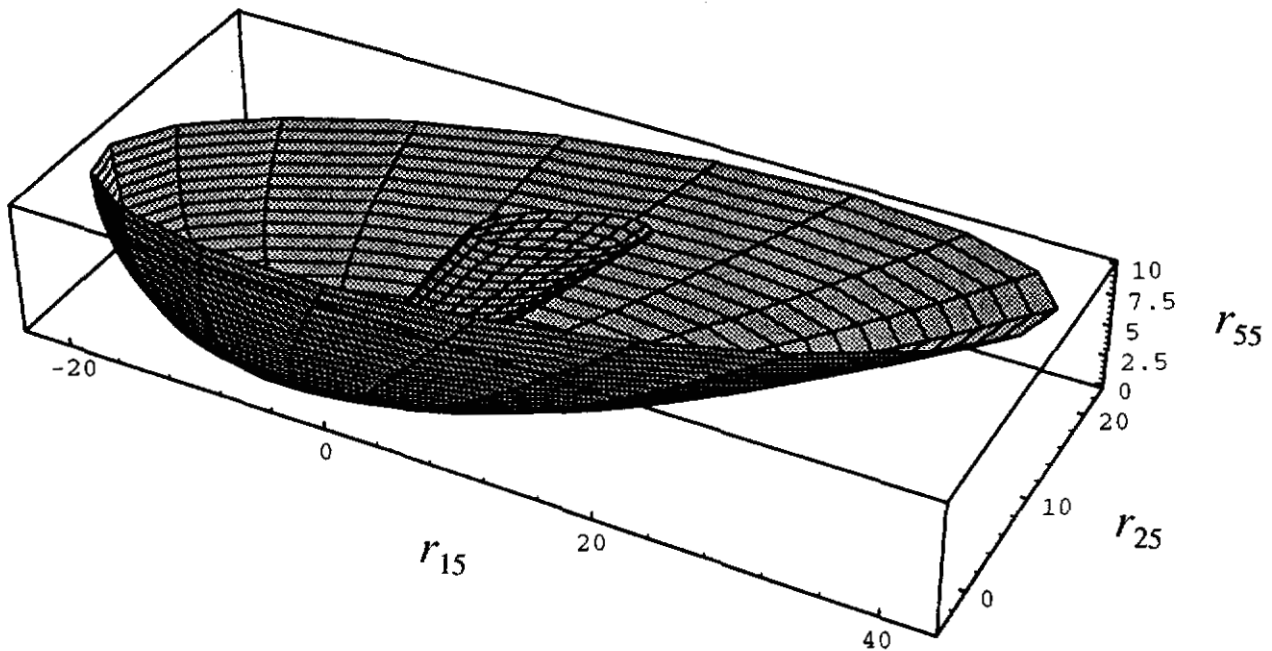

Fig.4: Domains $\mathcal{D}^i$ (external paraboloid) and $\overline{\mathcal{D}}$ (internal paraboloid) in the space described by r_{15}, r_{25}, and r_{55}.

QUESTIONS AND ANSWERS

Question: Is it possible for the domain to shrink to zero?

P. Tavella, Instituto Elettrotecnico Nazionale: Yes, in fact for example, I suppose that one correlation is possible. You just shrink to the region when all the covariances term are zero. You just have no more degrees of freedom.